Soft-Output MSE Decoding of Lattices

Catherine LAMY[†] - Joseph BOUTROS[‡]

[†]Motorola Semiconductors Toulouse, France

[‡]Communications and Electronics Dept, ENST, 46 rue Barrault 75013 Paris, France

lamyc@com.enst.fr, boutros@com.enst.fr

January 18, 1999

Abstract

We present a universal MSE algorithm for lattice decoding in dimensions up to 1024 for both Gaussian and Rayleigh fading channels. The soft-output decoding is performed by a decision feedback equalizer. The problem of selecting a good rotation is also considered, and we show that a random high dimensional rotation exhibits very good performance on a Rayleigh fading channel.

Key Words : Lattices, random rotations, diversity, decision feedback decoding.

1 Introduction and definitions

The maximum likelihood (ML) decoding of a lattice Λ is equivalent to minimizing the Euclidian distance $||\mathbf{r}-\mathbf{x}||^2$ between the observed point $\mathbf{r}$ and the lattice point $\mathbf{x}$. This general task is too complex so special decoding techniques have been developed for each lattice type on the Gaussian channel ([1]). The ML decoding on a coherent Rayleigh fading channel with real fading coefficients $\{\alpha_i\}$ is equivalent to the minimization of the metric $||\mathbf{r}-\alpha * \mathbf{x}||^2 = \sum_{i=0}^{n-1} |r_i - \alpha_i x_i|^2$.

Considering the system model presented in Fig. 1, the point error rate P_e on the Gaussian channel is given by ([2])

$$P_{e_{point}} \approx \frac{\tau}{2}\, erfc\left(\sqrt{\frac{3s}{2^{s+1}} \times \frac{E_b}{N_0} \times \gamma(\Lambda)}\right) \tag{1}$$

where s is the number of bits per two dimensions, τ the lattice kissing number, $\gamma(\Lambda) = \frac{d^2_{Emin}}{\sqrt[\frac{n}{2}]{vol(\Lambda)}}$ the fundamental gain, d_{Emin} the minimal Euclidian distance and $vol(\Lambda)$ the fundamental volume of the lattice Λ.

On the Rayleigh fading channel, the point error rate is given by ([2])

$$P_{e_{point}} \leq \sum_{l=L}^{n} \frac{K_l}{\left(\frac{s}{8}\frac{E_b}{N_0}\right)^l} \tag{2}$$

where the positive constants K_l depend on the choice of Λ. L is the lattice diversity order and is the minimum number of distinct components between any two points of Λ, $L = \min_{\forall \mathbf{x},\mathbf{y}\in\Lambda} d_H(\mathbf{x},\mathbf{y})$.

Maximizing the diversity is the best way to decrease the error probability on the Rayleigh fading channel, and can be done by using a multidimensional rotation on the lattice, as illustrated in Fig. 2 on a 4-PSK. Rotations do not affect the lattice fundamental gain due to packing density, therefore they can be used on the Gaussian channel without any loss in performance. One drawback is that they can not be decoded using Gaussian channel algorithms mentioned above nor with the ML algorithm (*Universal sphere decoder*) proposed in [4] as it is too complex for dimensions higher than 32.

In this paper, we generalize the equalizer presented in [5] and propose (see section 2) a sub-optimal universal algorithm for decoding an arbitrary lattice Λ in dimension up to 1024 for both Gaussian and Rayleigh fading channels. The excellent performance obtained in [5] are explained by studying the diversity distributions of rotated lattices (see section 3). Surprinsigly, we show that rotated lattices can be randomly chosen for the fading channel.

2 Minimum MSE decoding of lattices with DFE

Let us consider a lattice $\Lambda = M\mathbb{Z}^n$, i.e. a group in $\mathbb{R}^n$ or $\mathbb{C}^n$ obtained by a linear transform of the group $\mathbb{Z}^n$, with M the lattice generator matrix. The effect of this matrix on $\mathbb{Z}^n$ is similar to an ISI channel, as each component of a lattice point is a linear combination of all the input integers. Thus, decoding Λ is equivalent to suppressing ISI, operation that can be performed with an equalizer ([6]). Due to the dramatic complexity (for high dimension lattices) of trellis equalization used in ML equalization, we will use a sub-optimal MSE equalizer with decision feedback (Cf. Fig. 3).

The estimation of the i^{th} component of lattice point $\mathbf{x}$ is not used in the feedback equalization of the i^{th} received symbol, and so we impose the following condition

$$\forall i \in \{0, ...n-1\} \qquad | g_{ii} |= 0 \tag{3}$$

We assume that the variance per component of vector $\mathbf{z}$ is 1, and that $\mathrm{E}[\mathbf{z}\hat{\mathbf{z}}^h] = \rho I_n$ where ρ is a correlation factor and I_n is the identity matrix. Practically, ρ is approximated by $\rho \approx (1 - P_e(z_i))$, where $P_e(z_i)$ is the error probability on the integer component z_i. Hence, $\rho = 1$ when the error rate is too small. $\mathbf{z^t}$ denotes the transpose of $\mathbf{z}$, $\mathbf{z}^*$ the conjugate and $\mathbf{z^h}$ the transpose conjugate.

The DFE based on the MSE criterion minimizes $\mathrm{E}[||z - \tilde{z}||^2]$. Lagrange multipliers λ_i are used to take condition (3) into account. The minimization over W and G gives

$$\begin{cases} W^* = \frac{1}{N_0} D_{\rho\lambda + (1-\rho^2)} M^t V^* \quad \text{where V is defined by} \quad (\frac{1-\rho^2}{N_0} M^* M^t + I_n).V^* = I_n \\ \\ G^* = \frac{\rho}{N_0} D_{\rho\lambda + (1-\rho^2)} M^t V^* M^* + D_{\lambda - \rho} \end{cases} \tag{4}$$

where any D_ξ denote a diagonal matrix $\mathrm{Diag}(\xi_0 \ldots \xi_{n-1})$ and where the vector $(\lambda_0, \ldots, \lambda_{n-1})$ is noted λ. The Lagrange multipliers are given by the constraint on G, so we obtain finally

$$W = D_{\frac{1}{\rho^2 B^* + N_0}} M^h V \ and \ G = D_{\frac{\rho}{\rho^2 B^* + N_0}} M^h V M - D_{\frac{\rho B^*}{\rho^2 B^* + N_0}}$$

with $V^* = (v_{ij})$ $M = (m_{ij})$, $B = (B_0, \ldots, B_{n-1})$ and $B_i = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} m_{ki} v_{kl} m_{li}^*$

3 Fast Rotation Transform for Fading Channels

3.1 From algebraic rotations to Fast Rotations

The high diversity lattices presented in [2] are built by applying a canonical embedding to the ring of integers in a totally complex cyclotomic number field generated by $\theta = e^{2j\pi/N}$ ($n = \phi(N)$ where ϕ is the Euler function). The rotation matrix is given by the following $n/2 \times n/2$ complex form ([3])

$$R = (r_{ik}) \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \theta_1 & \theta_2 & \ldots & \theta_{n/2} \\ \vdots & \vdots & & \vdots \\ \theta_1^{n/2-1} & \theta_2^{n/2-1} & \ldots & \theta_{n/2}^{n/2-1} \quad \forall i \in \{1 \ldots n/2\} \quad \theta_i = \theta \times e^{4j\pi(i-1)/n} \end{pmatrix} \tag{5}$$

This $n/2$-dimensional complex rotation of diversity $n/2$ (full diversity) is equivalent to a real n-dimensional rotation of diversity $n/2$.

When n is a power of 2, $r_{ik} = (\theta . e^{\frac{4j\pi i}{n}})^k = e^{\frac{j\pi k}{n}} . e^{\frac{2j\pi ik}{n/2}} \qquad \forall (i,k) \in \{0 \ldots \frac{n}{2} - 1\}$.

Now, if this rotation is applied to a vector $X = (x_0, \ldots, x_{n/2-1})$, the result is the vector $P = (p_0, \ldots, p_{n/2-1})$ given by

$$p_i = \sum_{k=0}^{\frac{n}{2}-1} (r_{ik}.x_k) = \sum_{k=0}^{\frac{n}{2}-1} (e^{\frac{2j\pi ik}{n/2}}.x'_k) \qquad \text{with} \quad x'_k = x_k . e^{\frac{j\pi k}{2(n/2)}}. \tag{6}$$

If we compare (6) with the Discrete Fourier Transform (DFT), we see that an algebraic rotation based on the lattice $\mathbb{Z}_{n,n/2}$ (rotated version of $\mathbb{Z}^n$ with diversity $\frac{n}{2}$) is equivalent to $n/2$ different phase shifts followed by a DFT. Thus, by combining 2-dimensional rotations and a Fast Fourier Transform, we obtain a new type of fast transform : a complex Fast Rotation Transform (FRT).

3.2 Diversity distribution of random Hadamard-like matrices

Let us consider a spectral efficiency of 1 bit per dimension ($s = 2$). As an exact formula of the entries h_{ij} of a Hadamard matrix is very difficult to express, we define a random Hadamard-like matrix $H_n = (h_{ij})$

- the first row and the first column of H_n are filled with *'1'*.
- the other rows of H_n are composed by $n/2$ *'1'* (including the one in the first column) and $n/2$ *'-1'* equally distributed.
- we voluntarily forget the scaling factor $\dfrac{1}{\sqrt{n}}$.

Let us also define

- $\mathbf{u} = (u_i)$ the input vector with $\forall i, u_i \in \{0,1\}$ as $s = 2$, and $\mathbf{v} = (v_i)$ the ouput vector ($\mathbf{v} \in \Lambda$), such that $\mathbf{v} = H\mathbf{u}$.

  Note that $v_0 \neq 0$ for any $\mathbf{u} \neq \mathbf{0}$.
- $U_{k|0}$ the set of vectors $\mathbf{u}$ with k components equal to *'1'* and $u_0 = 0$.
- $U_{k|1}$ the set of vectors $\mathbf{u}$ with k components equal to *'1'* and $u_0 = 1$.
- For U being one of these two sets, we have

$$P(v_i = 0 \mid \mathbf{u} \in U) = P(v_j = 0 \mid \mathbf{u} \in U) = P_U(0) \quad \forall (i,j) \in \{1, ..., n-1\}^2 \tag{7}$$

- $L[l]$ is the number of vectors with diversity l.

This gives us the general formula of the diversity distribution of these random matrices :

$$\forall l \in \{1, ..., n-1\}$$

$$L[l] = \sum_{k'_1=1}^{n/2} \left(\binom{n-1}{2k'_1} P_{U_{2k'_1|0}}(0)^{n-l}(1 - P_{U_{2k'_1|0}}(0))^{l-1} + \binom{n-1}{2k'_1 - 1} P_{U_{2k'_1|1}}(0)^{n-l}(1 - P_{U_{2k'_1|1}}(0))^{l-1} \right)$$

and

$$L[n] = \sum_{k'_1=1}^{n/2} \left(\binom{n-1}{2k'_1} (1 - P_{U_{2k'_1|0}}(0))^{n-1} + \binom{n-1}{2k'_1 - 1} (1 - P_{U_{2k'_1|1}}(0))^{n-1} + \binom{n}{2k'_1 - 1} \right) \tag{8}$$

Similar expressions can be derived for higher spectral efficiency or the energy distribution.

4 Results

We present the results of DFE decoding of the rotated lattice $\mathbb{Z}_{512,256}$ obtained by a FRT in dimension 256 on a Rayleigh fading channel and the diversity distributions of the three fast transforms FHT, FFT and FRT for dimensions 8 and 512.

Formula (2) proves that the diversity distribution of Λ has a direct impact on its performance over the Rayleigh fading channel. Let us compare different diversity distributions to estimate their performance. Since $\mathbf{0}$ belongs to Λ, the comparison can be done with the probabilities $P(l)$ that l components of $\mathbf{x} \in \Lambda$ randomly chosen are non zero.

The FRT diversity distribution is simply given by $P(l) = 0$ for $l = 0 \ldots n/2 - 1$ and $P(n/2) = 1$. The FHT (or any random $[\pm 1]$ matrix) diversity distribution is approached by formula (8). Fig. 4-a shows that the FRT has much better performance in small dimensions than the two other transforms. Yet, for large dimensions, Fig. 4-b shows the distributions of FFT and FHT approach the FRT's one (notice the change in the scale).

Thus, for n large enough, the three transforms have the same performance on the Rayleigh fading channel while the FRT is definitely better for small dimensions (e.g. $n \leq 32$).

Fig. 5 present the 256-dimensional FRT on the Rayleigh fading channel with MSE decoding and the corresponding QAM on the Gaussian channel. The fading effect has clearly been suppressed : the

Rayleigh fading channel is converted into a Gaussian channel, and the diversity distribution is good enough to compensate for the sub-optimality of the MSE criterion.

The DFE decoding on a Gaussian channel of a Barnes-Wall lattice BW_{256} in dimension 256 having a fondamental gain of 10.5 dB on a Gaussian channel is a task currently being performed.

5 Conclusions

The dramatic complexity of the ML decoding of lattices in large dimensions makes the sub-optimal MSE decoder very attractive. The optimization of the MSE decoder is done in the integer space instead of the lattice space. The performance of the MSE decoder is excellent on the Rayleigh fading channel because the sub-optimality is compensated by the high diversity order of the rotated lattice.

When the dimension is large enough, in the case of the Rayleigh channel, the rotated lattice can be chosen at random (see [7] on how to generate random rotations) and it performs as good as an algebraic rotation given by $\mathbb{Z}_{n,n/2}$.

References

[1] J. H. Conway, N. J. Sloane : *Sphere packings, lattices and groups*, 2nd ed., 1993, Springer-Verlag, New York.

[2] J. Boutros, E. Viterbo, C. Rastello, J. C. Belfiore: "Good lattice constellations for both Rayleigh fading and Gaussian channels," *IEEE Trans. on Inf. Theory*, vol. 42, no. 2, pp. 502–518, March 1996.

[3] J. Boutros, E. Viterbo: "Signal space diversity: a power and bandwidth efficient diversity technique for the fading channel," *IEEE Trans. on Information Theory*, vol. 44, no. 4, July 1998.

[4] E. Viterbo, J. Boutros : "A universal lattice code decoder for fading channels," to appear in the *IEEE Trans. on Inf. Theory.*

[5] M. Reinhardt, J. Lindner : "Transformation of a Rayleigh fading channel; into a set of parallel AWGN channels and its advantage for coded transmission," *Electronics Letters*, vol.31, pp. 2154-2155, Dec. 1995.

[6] J. G. Proakis : Digital Communications, third edition, *McGraw-Hill*, 1995.

[7] N. J. A. Sloane : "Encrypting by random rotations," *Eurocrypt*, 1983.

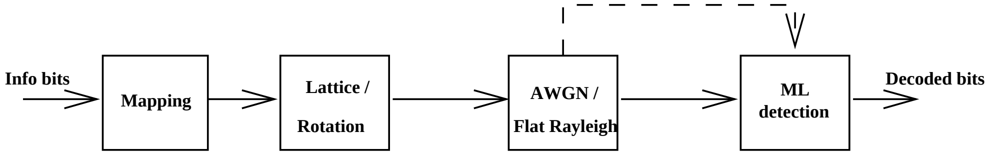


Figure 1: System model.

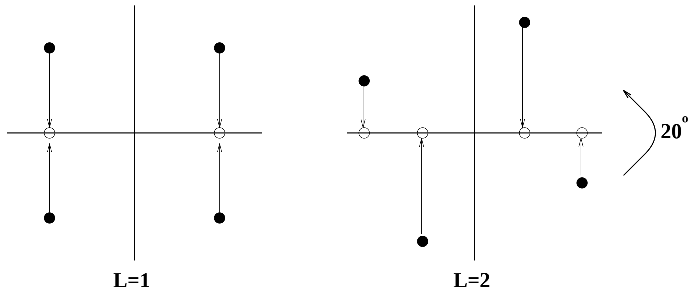


Figure 2: Increasing the diversity order by rotating the constellation.

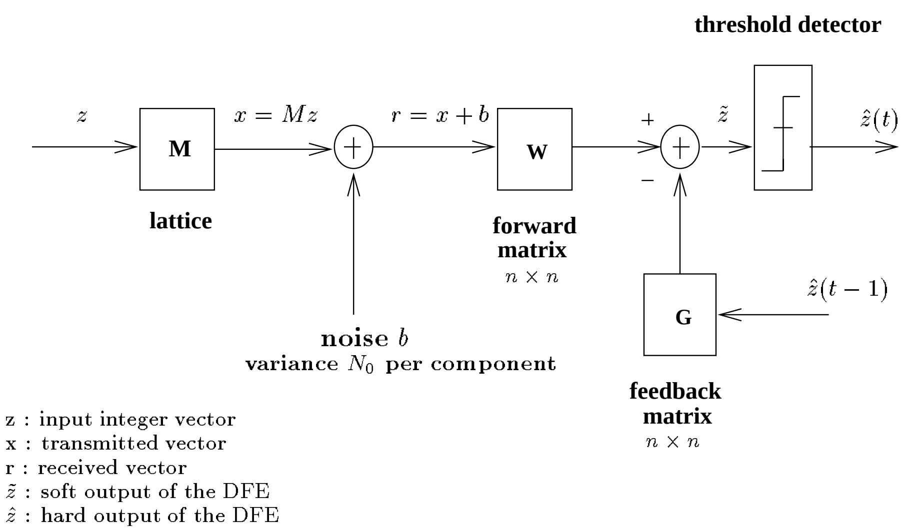
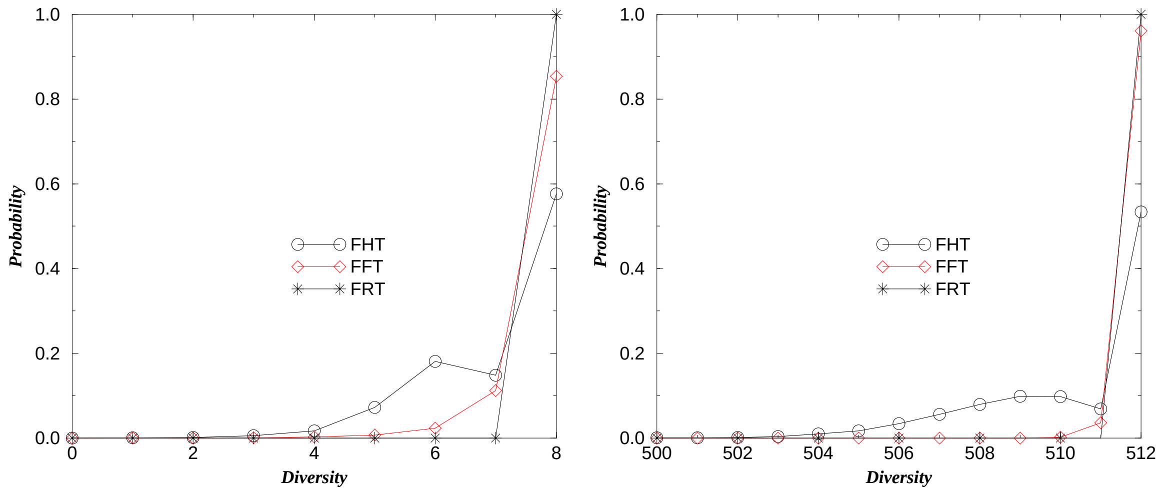
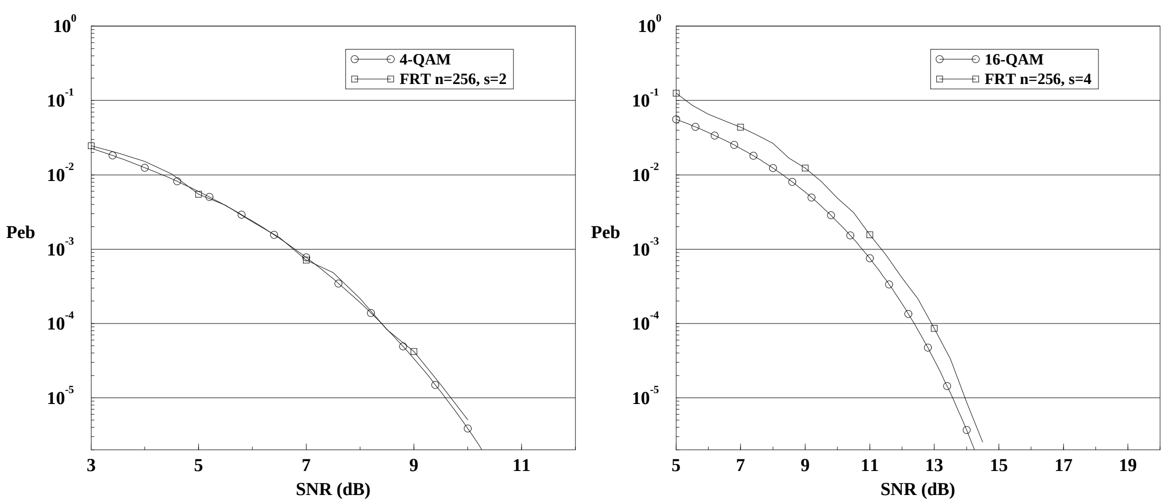


Figure 3: Decision feedback decoding of a lattice.



(a) **(b)**

Figure 4: Diversity distribution for 2 bits per symbol (a) : n=8, (b) : n=512.



(a) **(b)**

Figure 5: Binary error rate for an FRT with n=256. (a) : s=2, (b) : s=4.